PICARD HYPERBOLICITY OF MANIFOLDS ADMITTING NILPOTENT HARMONIC BUNDLES

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Abstract. For a quasi-compact Kähler manifold U endowed with a nilpotent harmonic bundle whose Higgs field is injective at one point, we prove that U is pseudo-algebraically hyperbolic, pseudo-Picard hyperbolic, and is of log general type. Moreover, we prove that there is a finite unramified cover $\tilde{U}$ of U from a quasi-projective manifold $\tilde{U}$ so that any projective compactification of $\tilde{U}$ is pseudo-algebraically hyperbolic, pseudo-Picard hyperbolic and is of general type. As a byproduct, we establish some criterion of pseudo-Picard hyperbolicity and pseudo-algebraic hyperbolicity for quasi-compact Kähler manifolds.

Contents

0. Introduction

0.1. **Main results.** The notion of Picard hyperbolicity for quasi-compact Kähler manifolds, which was introduced in [JK18, Den20b], is motivated by the classical big Picard theorem, which states that a holomorphic map $\Delta^* \to \mathbb{P}^1 \backslash \{0, 1\infty\}$ extends as holomorphic map to the whole disk Δ. Complex manifolds sharing this property with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ are then said to be *Picard hyperbolic*. This notion turns out to be an important hyperbolicity property since it implies the algebraicity of analytic maps from quasi-projective manifolds to Picard hyperbolic ones; this was first proven in [JK18]. The study of Picard hyperbolicity continues to have interesting developments: see e.g. the work of He–Ru [HR21] where a quantitative version is introduced, or Etesse [Ete20], who introduces a notion of intermediate Picard hyperbolicity, and gives applications to finiteness properties of automorphism groups.

In [Den20b], the second named author proved the Picard hyperbolicity for quasi-compact Kähler manifolds admitting a complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short) whose period map has zero dimensional fibers. $\mathbb{C}$-PVHS is a subcategory of *nilpotent harmonic bundles*. Our goal of this paper is to extend the results

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in [Den20b] to manifolds admitting nilpotent harmonic bundles. The first result is the following.

Theorem A. *Let U be a quasi-compact Kähler manifold. Assume that there is a nilpotent harmonic bundle (E,θ,h) over U so that $\theta : T_U \to \mathrm{End}(E)$ is injective at one point. Then U is pseudo-Picard hyperbolic, pseudo-algebraically hyperbolic, and is of log general type. Moreover, U can be equipped with a* unique *algebraic structure that makes it quasi-projective, and any* dominant meromorphic *map from another complex quasi-projective manifold to U is* algebraic.

See Definitions 1.3, 1.4 and 1.6 for definitions of nilpotent harmonic bundles, pseudo-algebraic hyperbolicity and pseudo-Picard hyperbolicity. Using ideas of [Den20b], we can prove a stronger result on the hyperbolicity of compactifications after taking finite unramified cover of U, which is the main result of this paper.

Theorem B ($\subset$Theorem 6.1)**.** *Let U be a quasi-compact Kähler manifold. Assume that there is a nilpotent harmonic bundle (E,θ,h) on U so that $\theta : T_U \to \mathrm{End}(E)$ is injective at one point. Then there is a finite unramified cover $\tilde{U} \to U$ from a quasi-projective manifold $\tilde{U}$ so that any smooth projective compactification X of $\tilde{U}$ is of general type, pseudo-algebraically hyperbolic and pseudo-Picard hyperbolic.*

The proofs of Theorems A and B both rely on some new criterion of pseudo-Picard hyperbolicity and pseudo-algebraic hyperbolicity for *quasi-compact Kähler manifolds*, which is a novelty of this paper.

Theorem C ($\subset$Theorem 3.1+Theorem 4.1)**.** *Let Y be a compact Kähler manifold, and let D be a simple normal crossing divisor on Y. Assume that $U := Y - D$ is equipped with a pseudo-Kähler metric ω whose holomorphic sectional curvature is bounded from above by a negative constant $-2\pi c$, then*

(i) *U is pseudo-algebraically hyperbolic and pseudo-Picard hyperbolic.*
(ii) *If the $(1,1)$ cohomology class $c\{\varpi\} - \{D\}$ is big, then Y is pseudo-Picard hyperbolic and pseudo-algebraically hyperbolic. Here ϖ is the closed positive $(1,1)$-current on Y which is the trivial extension of ω.*

0.2. **Related works.** After the work [JK18], Picard hyperbolicity drew a lot of attention over the last years. In [BBT18], the authors proved the algebraicity of analytic maps from a quasi-projective manifold to another one admitting a quasi-finite period map. In [DLSZ19] the second named author with Lu, Sun and Zuo proved the Picard hyperbolicity for moduli of polarized manifolds with semi-ample canonical sheaf. In [Den20b], the second named author proved Theorems A and B when the nilpotent harmonic bundle (E,θ,h) is moreover a complex polarized variation of Hodge structures. Similar results were later also obtained by Brotbek-Brunebarbe [BB20] and Brunebarbe [Bru20b]. Indeed, this paper is strongly inspired by the works [Den20b, BB20, Bru20b]: the proof of Theorem 3.1 is inspired by the Second Main Theorem of Brotbek-Brunebarbe [BB20], especially by their study of Nevanlinna characteristic function relative to positive currents; the proof of Theorem B follows the same line as that of [Den20b, Theorem B]. Though the methods in [Den20b] and [BB20, Bru20b] are quite different, a common ingredient is the use of Griffiths line bundle for systems of Hodge bundles, which has a nice global positivity property. For nilpotent harmonic bundles, we do not have such analogous algebraic objects, and thus the methods and results in [Den20b, BB20] cannot be applied directly. A novelty in this paper is the use of a transcendental cohomology big class $\{\varpi\}$ (see Theorem 6.1 for the precise definition) which plays a similar role in the proof as the Griffiths line bundle for complex variation of Hodge structures. This also enables us to simplify previous work [Den20b, BB20, Bru20b] since we do not use deep results in Hodge theory such as Schmid's nilpotent orbit theorems and Hodge norm estimates etc.

In [DLSZ19] the second named author with Lu, Sun and Zuo obtained the criterion for Picard hyperbolicity in terms of a *Finsler metric* for $T_Y(-\log D)$ with a *stronger* curvature property than the negativity of holomorphic sectional curvature. Indeed, we are not sure that our new criterion for Picard hyperbolicity Theorem 3.1 still holds if ω is only assumed to a $(1,1)$-*hermitian form*; its global positivity is crucial in the proof.

Theorem 6.1 is a generalization of several earlier work, stemming from the seminal paper of Mumford [Mum77], who proved that given an arithmetic lattice on a bounded symmetric domain, then all compactifications of quotients by sublattices of sufficiently high index are of general type. It was then shown later by the work of Brunebarbe [Bru20a], Rousseau [Rou16], the first named author [Cad18] that these compactification satisfy very strong algebraic or hyperbolicity properties. These results were later extended to varieties supporting variations of Hodge structures in [Den20b, Theorem B] and [BB20, Bru20b]; Theorem 6.1 can be seen as a generalization of these last results for varieties supporting nilpotent harmonic bundles.

After the completion of this paper, Yohan Brunebarbe informed us that they were able to prove that the quasi-projective manifold U in Theorem A is of log general type in an ongoing work with Jeremy Daniel towards the Shafarevich conjecture for open varieties.

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Notations

- A complex manifold is called *quasi-compact Kähler* if it is the Zariski dense open set of a compact Kähler manifold.
- For two real functions f and g on a complex manifold, we write $f \gtrsim g$ or $g \lesssim g$ if $f \geq \varepsilon g$ for some constant $\varepsilon > 0$.
- A *compact Kähler log pair* (resp. *projective log pair*) (Y, D) consists of a compact Kähler (resp. *projective*) manifold Y and a simple normal crossing divisor D on Y.
- A map $\mu : (X, \tilde{D}) \to (Y, D)$ between compact Kähler log pairs is called a *log morphism* if $\mu : X \to Y$ is a holomorphic map with $\tilde{D} \subset \mu^{-1}(D)$.
- The unit disk is denoted by Δ and Δ^* denotes the punctured unit disk.
- For any closed positive $(1,1)$ current T on a compact Kähler manifold, we write $\{T\}$ for its cohomology class. For two cohomology $(1,1)$ class α and β, we write $\alpha \geq \beta$ if $\alpha - \beta$ is pseudo effective.
- For a line bundle L with a singular hermitian metric h, its curvature current is denoted by $\Theta_h(L) := -\mathrm{dd^c} \log h$, where $\mathrm{dd^c} := \frac{i}{2\pi}\partial\bar{\partial}$.

1. Technical preliminaries

In this section we first recall the definitions of nilpotent harmonic bundles and algebraic hyperbolicity. We then state and prove some results on Picard hyperbolicity and closed positive $(1,1)$ currents, which will be used throughout this paper.

1.1. Harmonic bundles.

Definition 1.1 (Higgs bundle)**.** A Higgs bundle on a complex manifold Y is a pair (E, θ) consisting of a holomorphic vector bundle E on Y and an $\mathcal{O}_Y$-linear map

$$\theta : E \to E \otimes \Omega_Y^1$$

so that $\theta \wedge \theta = 0$. The map θ is called the *Higgs field*.

Definition 1.2 (Harmonic bundle)**.** A *harmonic bundle* (E, θ, h) consists of a Higgs bundle (E, θ) and a hermitian metric h for E so that the connection

$$D := D_h + \theta + \theta_h^*$$

is flat. Here D_h is the Chern connection of (E, h), and $\theta_h^* \in C^\infty(Y, \mathrm{End}(E) \otimes \Omega_Y^{0,1})$ is the adjoint of θ with respect to h.

Definition 1.3 (Nilpotent harmonic bundle)**.** A harmonic bundle (E, θ, h) is called *nilpotent* if the characteristic polynomial $\det(t - \theta) = t^{\mathrm{rank}\, E}$.

Note that a complex polarized variation of Hodge structures induces a nilpotent harmonic bundle.

1.2. **Algebraic hyperbolicity.** Algebraic hyperbolicity for a compact complex manifold was introduced by Demailly in [Dem97a, Definition 2.2]. He proved in [Dem97a, Theorem 2.1] that a compact complex manifold is algebraically hyperbolic if it is Kobayashi hyperbolic. The notion of algebraic hyperbolicity was generalized to log pairs by Chen [Che04].

Definition 1.4 (Pseudo-algebraic hyperbolicity)**.** Let (Y, ω_Y) be a compact Kähler manifold and let D be a simple normal crossing divisor on Y. For any reduced irreducible curve $C \subset Y$ such that $C \not\subset D$, we denote by $i_Y(C, D)$ the number of distinct points in the set $\nu^{-1}(D)$, where $\nu : \tilde{C} \to C$ is the normalization of C. Assume that $Z \subsetneq Y$ is Zariski closed proper subset of Y. If there is $\varepsilon > 0$ so that

$$2g(\tilde{C}) - 2 + i_Y(C, D) \geq \varepsilon \deg_{\omega_Y} C := \varepsilon \int_C \omega_Y$$

for all reduced irreducible curve $C \subset Y$ not contained in $Z \cup D$, (Y, D) is called *algebraically hyperbolic modulo* Z, and *pseudo-algebraically hyperbolic*. If $Z = \varnothing$, (Y, D) is called *algebraically hyperbolic*.

Note that the number $2g(\tilde{C}) - 2 + i_Y(C, D)$ depends only on the intersection of C with the complement $Y - D$. Hence the above notion of hyperbolicity also makes sense for quasi-projective manifolds: we say that a quasi-projective manifold U is algebraically hyperbolic if it has a log compactification (Y, D) which is algebraically hyperbolic.

However, it is unclear to us if Demailly's theorem extends to the non-compact case, i.e. if Kobayashi hyperbolicity, or Picard hyperbolicity, of $Y - D$ will imply the algebraic hyperbolicity of (Y, D). Note that Pacienza-Rousseau [PR07] have proved that if $Y - D$ is hyperbolically embedded into Y, the log pair (Y, D) (and thus $Y - D$) is algebraically hyperbolic.

1.3. **Picard hyperbolicity.** Let us first recall the definition of Picard hyperbolicity introduced in [Den20b]. We start with the following definition of admissible coordinate systems which will be used frequently.

Definition 1.5. (Admissible coordinates) Let Y be an n-dimensional complex manifold, and let D be a simple normal crossing divisor. Let p be a point of Y, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ are the components of D containing p. An *admissible coordinate system* around p is a tuple $(\Omega; z_1, \ldots, z_n; \varphi)$ (or simply $(\Omega; z_1, \ldots, z_n)$ if no confusion arises) where

- Ω is an open subset of Y containing p.
- φ is a holomorphic isomorphism $\varphi : \Omega \to \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \ldots, \ell$.

Definition 1.6 (pseudo-Picard hyperbolicity)**.** Let U be a quasi-compact Kähler manifold, and let Y be a smooth Kähler compactification. U is called *pseudo-Picard hyperbolic* if there is a Zariski closed proper subset $Z \subsetneq U$ so that any holomorphic map $f : \Delta^* \to U$ with $f(\Delta^*) \not\subset Z$ extends to a holomorphic map $\bar{f} : \Delta \to Y$. We also say that U is *Picard hyperbolic modulo* Z. If $Z = \varnothing$, U is simply called *Picard hyperbolic*.

In [Den20b, Lemma 4.3] we proved that Definition 1.6 does not depend on the compactification of U when $Z = \varnothing$. The proof of this statement is based on the deep extension theorem of meromorphic maps by Siu [Siu75], and is also valid when Z is not empty. Let us now give some interesting properties of pseudo-Picard hyperbolic manifolds, which generalizes [Den20b, Lemma 4.3].

Proposition 1.7. *Let U, Y be as in Definition 1.6, and assume that U is pseudo-Picard hyperbolic. Let X be a compact complex manifold and let D be a simple normal crossing divisor on X. If there is a meromorphic map $f : X - D \dashrightarrow U$ which is* dominant, *that is, its image contains a non-empty open set of U, then f extends to a meromorphic map $X \dashrightarrow Y$. In particular,*

(i) *any compact complex manifold containing U as a Zariski dense open set is bimeromorphic to Y.*
(ii) *the pseudo-Picard hyperbolicity of U in Definition 1.6 does not depend on the compactification Y.*

Proof. Write $V := X - D$. To prove that f extends to a meromorphic map $X \dashrightarrow Y$, it suffices to check that locally around D. By [Siu75, Theorem 1], any meromorphic map from a Zariski open set W° of a complex manifold W to a compact Kähler manifold Y extends to a meromorphic map from W to Y provided that the codimension of $W - W^\circ$ is at least 2. It then suffices to consider the extensibility of f around smooth points on D. Pick any such point $x \in D$ and choose admissible coordinates $(\Omega; x_1, \ldots, x_n)$ around x so that $\Omega \cap D = (x_1 = 0)$. The theorem follows if we can prove that $f : \Delta^* \times \Delta^{n-1} \dashrightarrow U$ extends to a meromorphic map $\Delta^{n+1} \dashrightarrow Y$. Let Z be the Zariski closed proper subset of Y as in Definition 1.6. Denote by S the subvariety of $\Delta^* \times \Delta^{n-1}$ of codimension at least two so that f is not a holomorphic map. Since f is assumed to be dominant, there is thus a dense open set $W \subset \Delta^{n-1}$ so that for any $z \in W$, $\Delta^* \times \{z\} \not\subset S$ and $f(\Delta^* \times \{z\} - S) \not\subset Z$. Then the restriction $f|_{:\Delta^* \times \{z\}} : \Delta^* \times \{z\} \dashrightarrow U$ is well-defined, which is moreover holomorphic. Since U is Picard hyperbolic modulo Z, $f : \Delta^* \times \{z\} \to U$ then extends to $\Delta \times \{z\} \to Y$ for $z \in W$. It then follows from [Siu75, p.442, $(*)$] that f extends to a meromorphic map $\overline{f} : \Delta^n \dashrightarrow Y$. We thus can conclude that $f : X - D \dashrightarrow U$ extends to a meromorphic map $X \dashrightarrow Y$.

Let Y' be another compact complex manifold containing U as a Zariski dense open set. We can apply the Hironaka theorem on resolution of singularities to assume that $Y' - U$ is a simple normal crossing divisor. By the above result, the identity map of U extends to a meromorphic map $Y' \dashrightarrow Y$ which is thus bimeromorphic. The second statement follows, which also implies the last claim. □

1.4. **Closed positive $(1,1)$-currents.** In this subsection we first recall some results concerning closed positive $(1,1)$-currents (see [Dem12a]). We then prove Lemma 1.15 which will be crucial in the proofs of our main results.

Definition 1.8 (Pseudo-Kähler metric)**.** Let X be a complex manifold. A $(1,1)$-form ω on X is called a *pseudo-Kähler metric* (or *pseudo-Kähler form*) if $d\omega = 0$, ω is semipositive, and strictly positive on a Zariski open set of X.

Definition 1.9. Let (X, ω) be a compact Kähler manifold. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a cohomology $(1,1)$-class of X. The class α is *nef* if for any $\varepsilon > 0$ there is a smooth closed

$(1,1)$-form $\eta_\varepsilon \in \alpha$ so that $\eta_\varepsilon \geq -\varepsilon\omega$. The class α is *pseudo-effective* if there is a closed positive $(1,1)$-current $T \in \alpha$. α is called *big* if there is a closed positive $(1,1)$-current $T \in \alpha$ so that $T \geq \delta\omega$ for some $\delta > 0$. Such a current T will be called a *Kähler current.*

For two cohomology $(1,1)$ classes α and β, we write $\alpha \geq \beta$ if $\alpha - \beta$ is a pseudo-effective class.

Boucksom's criterion [Bou02] asserts that a class α is big if there is a closed positive current $T \in \alpha$ so that $\int_X (T_{\rm ac})^{\dim X} > 0$, where $T_{\rm ac}$ denotes the absolutely continuous part of T with respect to any smooth measure on X.

The *non-Kähler locus* $E_{nK}(\alpha)$ of a big class α introduced by Boucksom [Bou04] measures how far α is from being Kähler. It is the transcendental generalization of the *augmented base locus* for big line bundles.

Definition 1.10 (non-Kähler locus)**.** Let X be a compact Kähler manifold and let α be a big class on X. The *non-Kähler locus* $E_{nK}(\alpha)$ of α is

$$E_{nK}(\alpha) := \bigcap_{T \in \alpha} \mathrm{Sing}(T),$$

where the intersection ranges over all Kähler currents $T \in \alpha$, and $\mathrm{Sing}(T)$ is the complement of the set of points $x \in X$ such that T is smooth around x.

Let us quote the following result by Boucksom [Bou04, Theorem 3.17].

Theorem 1.11 (Boucksom)**.** *Let α be a big class on a compact Kähler manifold. Then its non-Kähler locus $E_{nK}(\alpha)$ is a proper analytic subvariety. Moreover, there is a Kähler current $T \in \alpha$ with analytic singularities which is smooth outside $E_{nK}(\alpha)$.* □

If the class α is big and nef, in [CT15] Collins-Tosatti proved the following theorem on the characterization of its non-Kähler locus $E_{nK}(\alpha)$. It is a transcendental generalization of the Nakamaye theorem.

Theorem 1.12 (Collins-Tosatti)**.** *Let (X,ω) be a compact Kähler manifold. Let α be a big and nef $(1,1)$ class on X. Then*

$$E_{nK}(\alpha) = \mathrm{Null}(\alpha) := \bigcup_{\int_Z \alpha^{\dim Z} = 0} Z \tag{1.1}$$

where the union is taken over all positive dimensional irreducible analytic subvarieties Z in Y. □

Let us recall the following extension theorem of Skoda (see *e.g.* [Dem12b, (2.4) Theorem]) which will be used frequently.

Theorem 1.13 (Skoda)**.** *Let X be a (not necessarily compact) complex manifold and let A be a closed analytic subset of X. Assume that T is a closed positive (p,p)-current defined on $X - A$ so that T has locally* finite mass *in a neighborhood of any point of A. Then the trivial extension of T, denoted by $\widetilde{T}$, is also a closed positive (p,p)-current on X.* □

Recall that $\widetilde{T}$ is defined as follows. For any smooth test $(n-1,n-1)$-form η, we let

$$\widetilde{T}(\eta) := \int_{X-A} T \wedge \eta. \tag{1.2}$$

In particular, Skoda's theorem implies the following result due to Bishop [Bis64, Theorem 3], which will be used to prove Lemma 3.3.

Theorem 1.14 (Bishop)**.** *Let X be a (not necessarily compact) complex manifold and let A be a closed analytic subset of X. Let V be an analytic subset of pure dimension p of $X - A$. Assume that V has locally finite volume near A. Then the topological closure $\overline{V}$ of V is an analytic subset of X.* □

The following result, which is a variant of in [Den20a, Lemma 5.4], will be crucial throughout this paper.

Lemma 1.15. *Let (Y,D) be a compact Kähler log pair. Let ω be a pseudo-Kähler form on $U := Y - D$ with holomorphic sectional curvature bounded from above by a negative constant. Then*

(i) *the trivial extension of ω, denoted by ϖ, is a closed positive current;*
(ii) *the cohomology class $\{\varpi\}$ is big and nef;*
(iii) *for any admissible coordinates $(\Omega; z_1, \ldots, z_n)$, the local potential ϕ of $\varpi = \mathrm{dd^c}\phi$ satisfies*

$$\phi \gtrsim -\log\Big(\prod_{j=1}^{\ell}(-\log|z_j|^2)\Big). \tag{1.3}$$

Proof. Pick any point $x \in D$, and choose admissible coordinates $(\Omega; z_1, \ldots, z_n)$ centered at x so that $D \cap \Omega = (z_1 \cdots z_\ell = 0)$. Since the holomorphic sectional curvature of ω is bounded from above by a negative constant, by Ahlfors-Schwarz lemma, we can use [Cad16, Proposition 3.1.2], which implies that there is a constant $\delta > 0$ so that

$$\frac{1}{\delta}\omega \le \omega_P := \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^2(\log|z_j|^2)^2} + \sum_{k=\ell+1}^{n} \frac{\sqrt{-1}dz_k \wedge d\bar{z}_k}{(1-|z_k|^2)^2}. \tag{1.4}$$

Consequently, the local mass of ω is bounded. By Skoda's theorem, its trivial extension ϖ is a closed positive current. Since $\int_{Y-D}\omega^n > 0$, by Boucksom's criterion, $\{\varpi\}$ is big.

Since ϕ is a closed positive $(1,1)$-current, there is a psh function ϕ on Ω so that $\mathrm{dd^c}\phi = \varpi$. Now, by the very definition of trivial extension (1.2), $\varpi \le \delta\omega_P$. Note that

$$\omega_P = -\mathrm{dd^c}\log\Big(\prod_{j=1}^{\ell}(-\log|z_j|^2) \cdot \prod_{k=\ell+1}^{n}(1-|z_k|^2)\Big).$$

Since $\delta\omega_P - \varpi \ge 0$, the function

$$-\delta\mathrm{dd^c}\log\Big(\prod_{j=1}^{\ell}(-\log|z_j|^2) \cdot \prod_{k=\ell+1}^{n}(1-|z_k|^2)\Big) - \phi$$

is thus a psh function, and as such, it is locally bounded from above. The inequality (1.3) then follows. Therefore, ϖ has zero Lelong numbers everywhere. By the regularization theorem for closed positive currents of Demailly (see [Dem92, Corollary 6.4]), the class $\{\varpi\}$ is nef. The lemma is proved. □

The following result due to Brunebarbe [Bru20b, Proposition 3.3] will be used to prove Theorem 6.1.(i). For completeness sake, we provide a proof here.

Lemma 1.16. *Let (Y,D) be a compact Kähler log-pair, and let ω be a pseudo-Kähler form on $U := Y-D$ so that it has non-positive holomorphic bisectional curvature and holomorphic sectional curvature bounded from above by a negative constant $-2\pi c$. Then the class $K_Y + D - c\{\varpi\}$ is pseudo-effective.*

Proof. Let U_0 be the Zariski open set of U so that ω is strictly positive definite. Since ω has non-positive holomorphic bisectional curvature and holomorphic sectional curvature bounded from above by a negative constant $-2\pi c$, one has

$$-\mathrm{Ric}(\omega) \ge 2\pi c\omega.$$

Let h_{K_Y+D} be the singular hermitian metric on $K_Y + D$ induced by ω. We will prove that its curvature current $\Theta_{h_{K_Y+D}}(K_Y + D)$ is positive.

Pick any point $x \in D$, and choose admissible coordinates $(\Omega; z_1, \ldots, z_n)$ centered at x so that $D \cap \Omega = (z_1 \cdots z_\ell = 0)$. Then for the local frame $\sigma := d\log z_1 \wedge \cdots d\log z_\ell \wedge dz_{\ell+1} \wedge \cdots \wedge dz_n$ of $K_Y + D|_\Omega$, one has

$$e^{-\varphi} := |\sigma|^2_{h_{K_Y+D}} = \frac{idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_n \wedge d\bar{z}_n}{|z_1 \cdots z_\ell|^2 \omega^n} \gtrsim \prod_{j=1}^{\ell}(-\log |z_i|^2)$$

where the last inequality follows from (1.4). Hence the local potential φ of $h_{K_Y+D} = e^{-\varphi}$ is always locally bounded. One the other hand, $\mathrm{dd^c}\varphi = -\frac{1}{2\pi}\mathrm{Ric}(\omega) \geq 0$ over the Zariski dense open set U_0. This implies that the curvature current $\Theta_{h_{K_Y+D}}(K_Y + D) = \mathrm{dd^c}\varphi$ is positive everywhere. On the other hand, since ϖ is the trivial extension of ω, one has thus

$$\Theta_{h_{K_Y+D}}(K_Y + D) \geq c\varpi. \tag{1.5}$$

The lemma follows. □

2. Pseudo-Kähler metrics induced by nilpotent harmonic bundles

In this section we prove that the nilpotent harmonic bundle on a complex manifold U induces a pseudo-Kähler metric with nice curvature properties similar to the case of period domains.

Proposition 2.1. *Assume that U is a complex manifold that supports a harmonic bundle (E, θ, h) so that $\theta : T_U \to \mathrm{End}(E)$ is injective at one point. Then U admits a pseudo-Kähler metric ω with non-positive holomorphic bisectional curvature. If (E, θ, h) is moreover nilpotent, then the holomorphic sectional curvature of ω is bounded from above by $-\frac{1}{4^{\mathrm{rank}\,E-1}}$.*

Proof. We define a metric h_U as the pullback metric of h by the map $T_U \xrightarrow{\theta} \mathrm{End}(E)$. This gives

$$h_U(\xi_1, \xi_2) := \langle \theta(\xi_1), \theta(\xi_2) \rangle_h$$

for any $\xi_1, \xi_2 \in T_U$. The fundamental $(1,1)$-form ω relative to h_U can thus be written as

$$\omega = -i\mathrm{tr}(\theta_h^* \wedge \theta), \tag{2.1}$$

which shows that $\omega \geq 0$. Since $\theta : T_U \to \mathrm{End}(E)$ is immersive at one point, ω is therefore strictly positive at a general point. Moreover,

$$d\omega = -id\mathrm{tr}(\theta_h^* \wedge \theta) = -i\mathrm{tr}(D_h\theta_h^* \wedge \theta) + i\mathrm{tr}(\theta_h^* \wedge D_h\theta)$$

where D_h is the Chern connection of (E, h). Note that $D_h\theta = 0 = D_h\theta_h^*$, hence $d\omega = 0$. Thus ω is a pseudo-Kähler form.

Let $p \in U$ so that $T_U \to \mathrm{End}(E)$ is injective. Pick local coordinates $(z_1, \ldots, z_n)$ centered at p, and set $\theta_i := \theta(\frac{\partial}{\partial z_i})$. Denote by θ_i^* the adjoint of θ_i with respect to h. Write R to be the curvature tensor of ω, and denote by $R^{\mathrm{End}(E)}$ the curvature tensor of $\mathrm{End}(E)$ induced by the harmonic metric h. By the curvature decreasing properties of subbundles, the holomorphic bisectional curvature in the direction $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial z_j}$ is

$$R_{i\bar{j}j\bar{i}} := \langle R_{i\bar{j}}(\frac{\partial}{\partial z_j}), \frac{\partial}{\partial z_i} \rangle_{h_U} \leq \langle R^{\mathrm{End}(E)}_{i\bar{j}}(\theta_j), \theta_i \rangle_h.$$

By the flatness of $D_h + \theta + \theta_h^*$, we have $R^E_{i\bar{j}} = -[\theta_i, \theta_j^*]$, so $R^{\mathrm{End}(E)}_{i\bar{j}}(\theta_k) = -[[\theta_i, \theta_j^*], \theta_k]$. This gives

$$\begin{aligned} R_{i\bar{j}j\bar{i}} &\leq -\langle [[\theta_i, \theta_j^*], \theta_j], \theta_i \rangle_h \\ &= -\mathrm{tr}([[\theta_i, \theta_j^*], \theta_j]\theta_i^*) \\ &= -\mathrm{tr}([\theta_i, \theta_j^*][\theta_j, \theta_i^*]) \\ &= -|[\theta_i, \theta_j^*]|^2 \leq 0. \end{aligned}$$

We conclude that ω has non-positive holomorphic bisectional curvature at any point $p \in U$ where θ is injective.

Assume now (E, θ, h) is moreover nilpotent. Then $\theta_i : E_x \to E_x$ is a nilpotent endomorphism for each i and $x \in U$. Recall that the holomorphic sectional curvature in the direction $\frac{\partial}{\partial z_i}$ is defined by

$$K(\frac{\partial}{\partial z_i}) := \frac{R_{i\bar{i}i\bar{i}}}{|\frac{\partial}{\partial z_i}|^4} \leq \frac{-|[\theta_i, \theta_i^*]|^2}{|\theta_i|^4}$$

Since θ_i is nilpotent, by Lemma 2.2 below, one has

$$|[\theta_i, \theta_i^*]| \geq \frac{|\theta_i|^2}{2^{\mathrm{rank}\, E-1}}.$$

This proves that

$$K(\frac{\partial}{\partial z_i}) \leq -\frac{1}{4^{\mathrm{rank}\, E-1}}.$$

Since the local coordinate is arbitrary, this proves that the holomorphic sectional curvature of ω is bounded from above by $-\frac{1}{4^{\mathrm{rank}\, E-1}}$. □

The following lemma of linear algebra was outlined in [Sim92, p. 27].

Lemma 2.2. *Let A be a nilpotent $n \times n$-matrix with values in the complex numbers, and let H be an hermitian definite positive matrix of size n. Let $A^* := H^t\overline{A}H^{-1}$ be the adjoint of A with respect to H. Then $|[A, A^*]|_H \geq \frac{1}{2^{n-1}}|A|_H^2$, where $|A|_H^2 = \frac{1}{2^{n-1}}\mathrm{tr}(AA^*)$.*

Proof. Since A is nilpotent, there is a strictly decreasing flag $\mathbb{C}^n = F_0 \supsetneq F_1 \supsetneq \ldots \supsetneq F_m = 0$ $(m \leq n)$ and such that $AF_i \subset F_{i+1}$. Applying the standard orthonormalization process, we may assume that the flag (F_i) is adapted to a H-unitary base of $\mathbb{C}^n$. Changing the standard base to this new base, we may now assume that A is strictly upper triangular, and H is the identity.

Write $A := (a_{ij})_{1 \leq i,j \leq n}$. Denote by $[A, A^*] := (b_{ij})_{1 \leq i,j \leq n}$. Since A is strictly upper triangular, then

$$b_{ii} = \sum_{j=i+1}^{n} |a_{ij}|^2 - \sum_{k=1}^{i-1} |a_{1k}|^2.$$

Set $c_i := \sum_{j=i+1}^{n} |a_{ij}|^2$. Then $\sum_{i=1}^{n-1} c_i = |A|^2$. There exists an integer m with $1 \leq m \leq n-1$ so that

$$c_i < \frac{1}{2^{n-i}}|A|^2$$

for $i < m$ and

$$c_m \geq \frac{1}{2^{n-m}}|A|^2.$$

Note that $b_{ii} \geq c_i - \sum_{j=1}^{i-1} c_j$. This implies

$$b_{mm} \geq \frac{1}{2^{n-1}}|A|^2.$$

The lemma follows from the fact that $|[A, A^*]|^2 \geq |b_{mm}|^2$. □

3. Criterion for Picard hyperbolicity

In this section we will establish our criterion for pseudo-Picard hyperbolicity of *quasi-compact Kähler manifolds*. Theorems 3.1.(i) and 3.1.(ii) will be used to prove Theorems A and B respectively. Their proofs are inspired by the Second Main theorem of Brotbek-Brunebarbe [BB20] and by [Yam19, Lemma 5.1]. Since we work on Kähler manifolds rather than projective ones, we have to establish the criterion on removable singularities of holomorphic maps from punctured disks into compact Kähler manifold in term of the growth of Nevanlinna characteristic functions (see Lemma 3.3).

Theorem 3.1. *Let Y be a compact Kähler manifold, and let D be a simple normal crossing divisor on Y. Assume that $U := Y - D$ is equipped with a pseudo-Kähler metric ω whose holomorphic sectional curvature is bounded from above by a negative constant $-2\pi c$, then*

(i) *U is Picard hyperbolic modulo the non-Kähler locus $E_{nK}(\{\varpi\})$. Here ϖ is the closed positive $(1,1)$-current on Y which is the trivial extension of ω, and its cohomology class $\{\varpi\}$ is big. Moreover,*

$$E_{nK}(\{\varpi\}) \subset Y - \{y \in U \mid \omega \text{ is strictly positive at } y\}. \tag{3.1}$$

(ii) *If $c\{\varpi\} - \{D\}$ is a big class, then Y is Picard hyperbolic modulo $E_{nK}(c\{\varpi\} - \{D\}) \cup D$.*

Proof. Our first step will be to prove an inequality similar to the Arakelov-Nevanlinna inequality of [BB20, Theorem 4.1] (see (3.5)). The method of using a current with Poincaré singularities to define a first Nevanlinna characteristic function is essentially the same; the arguments can be explained quite shortly in our context so we will recall them for completeness.

For any $f : \Delta^* \to Y$ with $f(\Delta^*) \not\subset D$, write $f^*\omega = i\sigma(z)dz \wedge d\bar{z}$. Since ω has negative holomorphic sectional curvature, $\sigma(z) \in L^1_{\rm loc}(\Delta^*)$, and

$$\mathrm{dd^c} \log |f'|^2_\omega \geq c f^*\omega \tag{3.2}$$

outside $f^{-1}(D)$. Indeed, let $z_0 \in \Delta^*$ be so that $f(z_0) \in D$. By the Ahlfors-Schwarz lemma, around z_0 we have

$$\sigma(z) \lesssim \frac{1}{|z - z_0|^2(\log |z - z_0|^2)^2}.$$

If ϕ is a local potential for ω, this shows that $\log \sigma(z) - f^*\phi + \log |z - z_0|^2$ is locally bounded from above near z_0, and thus extends as a psh function on the whole disk. Applying the dd^c-operator, one gets the inequality of $(1,1)$-currents

$$\mathrm{dd^c} \log \sigma(z) \geq i\sigma(z)dz \wedge d\bar{z} - [f^{-1}D].$$

Here $f^{-1}D$ is the reduced divisor on Δ^*, and $[f^{-1}D]$ is the associated current. In other words,

$$\mathrm{dd^c} \log |f'|^2_\omega \geq c f^*\omega - [f^{-1}(D)]. \tag{3.3}$$

holds over the whole Δ^*.

We now change our model of the disk into $\Delta^* := \{z \in \mathbb{C} \mid 1 < |z| < \infty\}$ by taking $z \mapsto \frac{1}{z}$, and define a *Nevanlinna characteristic function*

$$T_{f,\omega}(r) := \int_2^r \frac{dt}{t} \int_{\Delta_{2,t}} f^*\omega$$

where $\Delta_{2,t} := \{z \in \Delta^* \mid 2 < |z| < t\}$.

By Jensen formula, one has

$$\int_2^r \frac{dt}{t}\int_{\Delta_{2,t}} \mathrm{dd^c}\log|f'|^2_\omega = \int_0^{2\pi}\log|f'(re^{i\theta})|_\omega \frac{d\theta}{2\pi} - \int_0^{2\pi}\log|f'(2e^{i\theta})|_\omega \frac{d\theta}{2\pi} - 2\log\frac{r}{2}\int_0^{2\pi}\frac{\partial \log|f'|_\omega(2e^{i\theta})}{\partial r}\frac{d\theta}{2\pi}. \tag{3.4}$$

Using concavity of log, we have

$$\int_0^{2\pi}\log|f'(re^{i\theta})|_\omega \frac{d\theta}{2\pi} \le \frac{1}{2}\log\int_0^{2\pi}|f'(re^{i\theta})|^2_\omega \frac{d\theta}{2\pi}.$$

Note that

$$\frac{1}{2\pi r}\frac{d}{dr}(r\frac{d}{dr}T_{f,\omega}(r)) = \int_0^{2\pi}|f'(re^{i\theta})|^2_\omega \frac{d\theta}{2\pi}.$$

Since $T_{f,\omega}(r)$ and $r\frac{d}{dr}T_{f,\omega}(r)$ are both monotone increasing functions, we apply Borel's lemma [NW14, Lemma 1.2.1] twice so that, for any $\delta > 0$ one has

$$\begin{aligned}
\log\Big(\frac{d}{dr}(r\frac{d}{dr}T_{f,\omega}(r))\Big) &\le (1+\delta)\log\Big(r\frac{d}{dr}T_{f,\omega}(r)\Big) && \|\\
&= (1+\delta)\log r + (1+\delta)\log\Big(\frac{d}{dr}T_{f,\omega}(r)\Big) && \|\\
&\le (1+\delta)\log r + (1+\delta)^2\log T_{f,\omega}(r) && \|
\end{aligned}$$

Here $\|$ means that the inequality holds outside a Borel set $E \subset (2,\infty)$ of finite Lebesgue measure. The above inequalities yield

$$\frac{1}{2}\log\int_0^{2\pi}|f'(re^{i\theta})|^2_\omega \frac{d\theta}{2\pi} \le \frac{(1+\delta)^2}{2}\log T_{f,\omega}(r) + \frac{\delta}{2}\log r - \frac{1}{2}\log(2\pi) \quad \|.$$

Putting this into (3.4), we get

$$\begin{aligned}
\int_2^r \frac{dt}{t}\int_{\Delta_{2,t}} \mathrm{dd^c}\log|f'|^2_\omega &\le \frac{(1+\delta)^2}{2}\log T_{f,\omega}(r) + \frac{\delta}{2}\log r - \frac{1}{2}\log(2\pi)\\
&- \int_0^{2\pi}\log|f'(2e^{i\theta})|_\omega \frac{d\theta}{2\pi} - 2\log\frac{r}{2}\int_0^{2\pi}\frac{\partial \log|f'|_\omega(2e^{i\theta})}{\partial r}\frac{d\theta}{2\pi}. \quad \|.
\end{aligned}$$

By (3.3), this implies the requested inequality

$$c_1\log T_{f,\omega}(r) + c_2\log r + c_3 \ge cT_{f,\omega}(r) - N^{[1]}_{f,D}(r) \quad \| \tag{3.5}$$

for some positive constants c_1, c_2, c_3. Here $N^{[1]}_{f,D}(r)$ is the truncated counting function defined by

$$N^{[1]}_{f,D}(r) := \int_2^r \frac{dt}{t}\int_{\Delta_{2,t}}[f^{-1}(D)].$$

Obviously, it is zero if f avoids D. Note that in this case, we would have $T_{f,\omega}(r) \le \log(r)$, and the requested extension of f follows directly from Lemma 3.3 below if ω were the restriction of a Kähler form in Y on U. Since ω is merely *pseudo-Kähler* and is only defined on U, one needs some additional work, which makes use of the condition that ω is closed.

By Lemma 1.15, the trivial extension of ω over Y, denoted by ϖ, is a closed positive current. Moreover, the cohomology class $\{\varpi\}$ is big. By Theorem 1.11 there is a Kähler current $S_2 \in \{\varpi\}$ which is a smooth Kähler form on $Y - E_{nK}(\{\varpi\})$. We also choose a smooth closed $(1,1)$-form $\eta \in \{\varpi\}$.

Claim 3.2. *Fix any smooth Kähler metric ω_Y over Y. For any $f : \Delta^* \to Y$ with $f(\Delta^*) \not\subset E_{nK}(\{\varpi\})$, there are positive constants c_i so that*

$$T_{f,\omega}(r) \geq T_{f,\eta}(r) - c_4 \log T_{f,\omega_Y}(r) - c_4 \log r - c_5 \quad \| \tag{3.6}$$

$$T_{f,\eta}(r) \geq c_6 T_{f,\omega_Y}(r) - c_7 \log r - c_8 \quad \| \tag{3.7}$$

$$T_{f,\omega_Y} \geq c_9 T_{f,\omega}(r) - c_{10} \log r - c_{11} \quad \| \tag{3.8}$$

Proof of Claim 3.2. Write $D = \sum_{i=1}^{\ell} D_i$. Set σ_i to be a section $H^0(Y, \mathcal{O}_Y(D_i))$ defining D_i, and pick a smooth metric h_i for $\mathcal{O}_Y(D_i)$. Since $\eta \in \{\varpi\}$, there is a quasi-psh function $\varphi \leq 0$ defined on Y so that $\eta = \varpi - \mathrm{dd}^c\varphi$. By (1.3), one has

$$\varphi \geq -\delta_1 \log(\prod_{i=1}^{\ell} \log^2 |\varepsilon \cdot \sigma_i|^2_{h_i})$$

for some $\delta_1 > 0$ and $\varepsilon > 0$. By Jensen's formula, one has

$$\begin{aligned} T_{f,\omega}(r) - T_{f,\eta}(r) &= \int_0^{2\pi} \varphi \circ f(re^{i\theta}) \frac{d\theta}{2\pi} - \int_0^{2\pi} \varphi \circ f(2e^{i\theta}) \frac{d\theta}{2\pi} - \log\frac{r}{2} \int_0^{2\pi} \frac{\partial \varphi \circ f(2e^{i\theta})}{\partial r} \frac{d\theta}{2\pi} \\ &\geq -\delta_1 \int_0^{2\pi} \log(\prod_{i=1}^{\ell} \log^2 |\varepsilon \cdot \sigma_i|^2_{h_i}) \circ f(re^{i\theta}) \frac{d\theta}{2\pi} - c_4 \log r - c_5 \end{aligned}$$

By the concavity of log, one has

$$-\int_0^{2\pi} \log(\prod_{i=1}^{\ell} \log^2 |\varepsilon \cdot \sigma_i|^2_{\cdot h_i}) \circ f(re^{i\theta}) \frac{d\theta}{2\pi} \geq -2 \sum_{i=1}^{\ell} \log \int_0^{2\pi} (-\log |\varepsilon \cdot \sigma_i|^2_{h_i}) \circ f(re^{i\theta}) \frac{d\theta}{2\pi}$$

Using Jensen formula again, one obtains

$$\int_0^{2\pi} (-\log |\varepsilon \cdot \sigma_i|^2_{h_i}) \circ f(re^{i\theta}) \frac{d\theta}{2\pi} \leq T_{f,\Theta_{h_i}(D_i)}(r) + O(\log r).$$

(3.6) follows from the fact that

$$T_{f,\Theta_{h_i}(D_i)}(r) \leq \delta_2 T_{f,\omega_Y}(r)$$

for some positive constants δ_2.

Since S_2 and η are both in $\{\varpi\}$, there is a quasi-psh function $\phi \leq 0$ defined on Y so that $\eta = S_2 - \mathrm{dd}^c\phi$. Since S_2 is smooth over $U_0 := Y - E_{nK}(\varpi)$, and $f(\Delta^*) \cap U_0 \neq \varnothing$, f^*S_2 is thus well defined on Δ^*. By Jensen formula again, so one has

(3.9)

$$\begin{aligned} T_{f,\eta}(r) - T_{f,S_2}(r) &= -\int_0^{2\pi} \phi \circ f(re^{i\theta}) \frac{d\theta}{2\pi} + \int_0^{2\pi} \phi \circ f(2e^{i\theta}) \frac{d\theta}{2\pi} - \log\frac{r}{2} \int_0^{2\pi} \frac{\partial \phi \circ f(2e^{i\theta})}{\partial r} \frac{d\theta}{2\pi} \\ &\geq -c_7 \log r - c_8. \end{aligned}$$

On the other hand, $S_2 \geq \varepsilon\omega_Y$ for some constant $\varepsilon > 0$ since S_2 is a Kähler current, one has

$$T_{f,S_2}(r) \geq \varepsilon T_{f,\omega_Y}(r).$$

This proves (3.7).

In a similar vein as in (3.9), one can prove that

$$T_{f,\eta}(r) - T_{f,\omega}(r) \geq -\delta_3 \log r - \delta_4.$$

Since $T_{f,\omega_Y}(r) \geq \delta_5 T_{f,\eta}(r)$, (3.8) follows. □

Let us prove Theorem 3.1.(i). For any $f : \Delta^* \to U$ with $f(\Delta^*) \not\subset E_{nK}(\{\varpi\})$, one has $N^{[1]}_{f,D}(r) = 0$. Putting (3.6) and (3.7) into (3.5), we immediately conclude that $T_{f,\omega_Y}(r) \sim \log r$ when $r \to \infty$. This proves that f extends across the point ∞ by Lemma 3.3 below, hence U is Picard hyperbolic modulo $E_{nK}(\{\varpi\})$.

Let us prove (3.1). By Lemma 1.15 $\{\varpi\}$ is big and nef. By Theorem 1.12, one has

$$E_{nK}(\{\varpi\}) = \mathrm{Null}(\{\varpi\}) := \bigcup_{\int_Z \{\varpi\}^{\dim Z}=0} Z$$

where the union is taken over all positive dimensional irreducible analytic subvarieties Z in Y. If

$$Z \not\subset Y - \{y \in U \mid \omega \text{ is strictly positive at } y\}.$$

by (1.3) one has

$$\int_Z \{\varpi\}^{\dim Z} = \int_{Z^{\mathrm{reg}}\cap U} \omega^{\dim Z} > 0.$$

This yields (3.1) by (1.12). (3.1) is proved.

Let us now prove Theorem 3.1.(ii). Since $c\{\varpi\} - \{D\}$ is big, by Theorem 1.11 one can take a Kähler current $S_3 \in \{c\varpi\} - \{D\}$ which is smooth outside the non-Kähler locus $E_{nK}(\{c\varpi\} - \{D\})$. Let $f : \Delta^* \to Y$ be a curve which is not contained in $E_{nK}(\{c\varpi\} - \{D\}) \cup D$. Since $\{S_3 + [D]\} = \{c\varpi\} = c\{\eta\}$, similar arguments as (3.9) show that

$$cT_{f,\eta}(r) - T_{f,S_3+[D]}(r) \geq -c_9 \log r - c_{10}.$$

Moreover,

$$T_{f,S_3+[D]}(r) = T_{f,S_3}(r) + T_{f,[D]}(r) \geq c_{11}T_{f,\omega_Y}(r) + N^{[1]}_{f,D}(r).$$

Combining these inequalities with (3.6), (3.5) and (3.8), we conclude that $T_{f,\omega_Y}(r) \sim \log r$. This proves that f extends across the point ∞ by Lemma 3.3 below. □

We state and prove the following criterion on the extendibility across the origin of the holomorphic map from the punctured disk to a *compact Kähler manifold*.

Lemma 3.3. *Let (Y, ω_Y) be a compact Kähler manifold, and let $f : \Delta^* \to Y$ be a holomorphic map from the punctured disk to Y. If*

$$T_{f,\omega_Y}(r) := \int_2^r \frac{dt}{t} \int_{\Delta_{2,t}} f^*\omega_Y$$

is bounded from above by $C \log r$ when $r \to \infty$ for some constant $C > 0$. Here we consider our model of the punctured disk as $\Delta^* := \{z \in \mathbb{C} \mid 1 < |z| < \infty\}$ by taking $z \mapsto \frac{1}{z}$. Then f extends to a holomorphic map $\Delta^* \cup \{\infty\} \to Y$.

Proof. We claim that $\int_{\Delta_{2,t}} f^*\omega_Y < 3C$ for any $t > 0$. Or else, there is $r_0 > 0$ so that $\int_{\Delta_{2,t}} f^*\omega_Y \geq 3C$ when $t \geq r_0$. Then

$$T_{f,\omega_Y}(r) \geq 3C(\log r - \log r_0) \geq 2C \log r$$

if $r \gg 0$. This contradicts with our assumption.

For simplicity, let us now change our model of the punctured disk to $\Delta^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ by taking $z \mapsto \frac{1}{z}$. Then one has

$$\int_{\{z\in\mathbb{C}|0<|z|<\frac{1}{2}\}} f^*\omega_Y < 3C.$$

Consider the graph V of f, which is an one dimensional closed analytic subvariety of $\Delta^* \times Y$. Let us equip $\Delta \times Y$ with the Kähler metric $\omega' := q_1^*\omega_e + q_2^*\omega_Y$, where $q_1 : \Delta \times Y \to \Delta$

and $q_2 : \Delta \times Y \to Y$ is the projection map, and $\omega_e := idz \wedge d\bar{z}$. Then the volume of the analytic set $V \cap \{z \mid 0 < |z| < \frac{1}{2}\} \times Y$ with respect to the Kähler metric ω' is

$$\int_{\{z \in \mathbb{C} | 0 < |z| < \frac{1}{2}\}} f^* \omega_Y + \omega_e \leq 3C + \pi.$$

We now apply Theorem 1.14 to conclude that the closure of V in $\Delta \times Y$, denoted by $\overline{V}$, is an one dimensional closed analytic subset. Hence the map $q_1|_{\overline{V}} : \overline{V} \to \Delta$ is a *proper* holomorphic map, which is an isomorphism over Δ^*. Therefore, $q_1|_{\overline{V}}$ is moreover an isomorphism. The composition $q_2 \circ (q_1|_{\overline{V}})^{-1} : \Delta \to Y$ is a holomorphic map which extends f. The proposition is proved. □

Remark 3.4. Note that Lemma 3.3 is a well-known result when Y is a projective manifold; see *e.g.* [Dem97b, 2.11. Cas «local »] or [Siu15, Lemma 6.5]. For their strategy of the proof, they use sufficiently many global rational functions on Y to reduce the theorem to holomorphic maps $\Delta^* \to \mathbb{P}^1$ and then apply Nevanlinna's logarithmic derivative lemma to conclude. Our proof of Lemma 3.3 thus also provides an alternative and simplified proof in the projective setting.

4. Criterion for algebraic hyperbolicity

In this section we will establish an algebraic analogue to Theorem 3.1.

Theorem 4.1. *Let (Y, D) be a compact Kähler log pair. Assume that $U := Y - D$ is equipped with a pseudo-Kähler metric ω whose holomorphic sectional curvature is bounded above by a negative constant $-2\pi c$, then*

(i) *U is algebraic hyperbolic modulo $E_{nK}(\{\varpi\})$, where ϖ is the closed positive $(1,1)$-current on Y which is the trivial extension of ω.*
(ii) *If $c\{\varpi\} - \{D\}$ is a big class, then Y is algebraically hyperbolic modulo $E_{nK}(c\{\varpi\} - \{D\}) \cup D$.*

Proof. By Lemma 1.15, we know that $\{\varpi\}$ is big. Let C be any irreducible reduced curve not contained in $D \cup E_{nK}(\{\varpi\})$. Set $\nu : \tilde{C} \to C$ to be the normalization. Write $\tilde{C}^\circ := \nu^{-1}(U)$, and denote by $P := \nu^{-1}(D)$ the reduced divisor on $\tilde{C}$. By (3.1), $\nu^*\omega$ is also a pseudo-Kähler metric on $\tilde{C}^\circ$. Since the holomorphic sectional curvature of ω is bounded from above by a negative constant $-c$, by the curvature decreasing property, the holomorphic sectional curvature of $\nu^*\omega$ is also bounded above by $-2\pi c$. As in the proof of Lemma 1.16, $\nu^*\omega$ induces a singular hermitian metric $h_{K_{\tilde{C}}+P}$ whose curvature current is positive. Moreover, by (1.5), one has

$$\Theta_{h_{K_{\tilde{C}}+P}}(K_{\tilde{C}} + P) \geq c\widetilde{\nu^*\omega}$$

where $\widetilde{\nu^*\omega}$ is the closed positive current on $\tilde{C}$ which is the trivial extension of $\nu^*\omega$. By (1.3), the Lelong numbers of the local potentials ϕ of ϖ are 0, so using $\nu^*\varpi \overset{\text{loc}}{=} dd^c(\phi \circ \nu)$, one can easily check that $\widetilde{\nu^*\omega} = \nu^*\varpi$. Hence

$$2g(\tilde{C}) - 2 + i_Y(C, D) = \int_{\tilde{C}} \Theta_{h_{K_{\tilde{C}}+P}}(K_{\tilde{C}} + P) \geq c \int_{\tilde{C}} \nu^*\varpi = c\{C\} \cdot \{\varpi\}, \tag{4.1}$$

where we use the notation in Definition 1.4.

Fix a Kähler form ω_Y on Y. By Theorem 1.11 one can choose a Kähler current $S_1 \in \{\varpi\}$ which is smooth outside $E_{nK}(\{\varpi\})$. Hence, there is a constant $\varepsilon > 0$ so that $S_1 \geq \varepsilon\omega_Y$. Since C is not contained in $E_{nK}(\{\varpi\}$, one has

$$\{C\} \cdot \{\varpi\} = \int_{\tilde{C}} \nu^* S_1 \geq \varepsilon \int_{\tilde{C}} \nu^*\omega_Y = \varepsilon \deg_{\omega_Y} C.$$

Putting this inequality into (4.1), we obtain

$$2g(\tilde{C}) - 2 + i_Y(C,D) \geq c\varepsilon \deg_{\omega_Y} C.$$

The first claim follows since $c > 0$ and $\varepsilon > 0$ does not depend on C.

If $c\{\varpi\} - \{D\}$ is big, by Theorem 1.11 again there is a Kähler current $S_2 \in c\{\varpi\} - \{D\}$ which is smooth outside $E_{nK}(c\{\varpi\} - \{D\})$. Hence there is a constant $\varepsilon_2 > 0$ so that $S_2 \geq \varepsilon_2 \omega_Y$. If C is not contained in $D \cup E_{nK}(c\{\varpi\} - \{D\})$, by $S_2 + [D] \in c\{\varpi\}$ one has

$$c\{C\} \cdot \{\varpi\} = \int_{\tilde{C}} \nu^*(S_2 + D) \geq \varepsilon_2 \int_{\tilde{C}} \nu^* \omega_Y + i_Y(C,D) = \varepsilon_2 \deg_{\omega_Y} C + i_Y(C,D).$$

Putting this to (4.1), we obtain

$$2g(\tilde{C}) - 2 \geq \varepsilon_2 \deg_{\omega_Y} C.$$

This proves the second claim. □

5. Proof of Theorem A

We are now ready to prove Theorem A.

Proof of Theorem A. Take a compact Kähler manifold Y compactifying U so that $D := Y - U$ is simple normal crossing. By Proposition 2.1, the nilpotent harmonic bundle induces a pseudo-Kähler metric ω on U whose holomorphic bisectional curvature is non-positive and holomorphic sectional curvature is bounded from above by $-\frac{1}{2^{\operatorname{rank} E-1}}$. One can then apply the criterion in [Cad16, Theorem 2] or [BC20, Theorem 1.6] to conclude that U is of log general type. Alternatively, by Lemma 1.16, $K_Y + D \geq \frac{1}{4^{\operatorname{rank} E-1} \cdot 2\pi}\{\varpi\}$ where ϖ is the closed positive current on Y which is the trivial extension of ω. Since $\{\varpi\}$ is big, $K_Y + D$ is also big. This also proves that U is of log general type. Hence Y is both a Kähler and Moishezon manifold, hence projective. By Proposition 1.7.(i), any compact complex manifold compactifying U is bimeromorphic to Y. This proves the uniqueness of algebraic structure of U by Chow's theorem.

It follows from Theorems 3.1.(i) and 4.1.(i) that U is pseudo-Picard hyperbolic and pseudo-algebraically hyperbolic. □

A direct consequence of Theorems 3.1.(i) and 4.1.(i) is the following result.

Corollary 5.1. *Let U be a quasi-projective manifold. If U is equipped with a Kähler metric ω with holomorphic sectional curvature bounded from above by a negative constant, then U is Picard hyperbolic and algebraically hyperbolic.* □

The above result gives a new proof of the following theorem by Borel [Bor72], Kobayashi-Ochiai [KO71] and Pacienza-Rousseau [PR07].

Theorem 5.2. *Let U be a quasi-projective quotient of bounded symmetric domain by a torsion free lattice. Then U is Picard hyperbolic and algebraically hyperbolic.*

Proof. Since the Bergman metric on U is Kähler with holomorphic sectional curvature bounded from above by a negative constant, the Picard hyperbolicity and algebraic hyperbolicity of U follows from the above corollary immediately. □

6. Hyperbolicity for the compactification after finite unramified cover

In this section we will prove Theorem B using ideas similar to [Den20b, Proof of Theorem 5.1].

Theorem 6.1. *Let (Y, D) be a compact Kähler log pair. Assume that there is a nilpotent harmonic bundle (E, θ, h) on $U := Y - D$ so that $\theta : T_U \to \mathrm{End}(E)$ is injective at one point. Then there is a log morphism $\mu : (X, \tilde{D}) \to (Y, D)$ from a projective log pair $(X, \tilde{D})$ which is a finite unramified cover over U such that*

(i) *any irreducible subvariety of X non contained in the analytic subvariety $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$ is of general type;*
(ii) *X is Picard hyperbolic modulo $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$;*
(iii) *X is algebraically hyperbolic modulo $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$.*

Here ϖ is the trivial extension of the pseudo-Kähler form $\omega = -i\mathrm{tr}(\theta_h^* \wedge \theta)$ on U defined in (2.1). Moreover, we have

$$E_{nK}(\{\varpi\}) \subset Y - \{y \in U \mid \theta \text{ is injective at } y\}. \tag{6.1}$$

We will need the following crucial result proved in [Den20b, Claim 5.2] to find the desired covering $\mu : X \to Y$ in Theorem 6.1. The proof is based on residual finiteness of the global monodromy group and Cauchy's argument theorem.

Lemma 6.2. *Let Y be a projective manifold and let $D = \sum_{j=1}^{\ell} D_j$ be a simple normal crossing divisor on Y. Assume that there is a complex local system $\mathcal{L}$ over $U := Y - D$. Then for any $m > 0$, there is a smooth projective log pair $(X, \tilde{D})$ and a log morphism $\mu : (X, \tilde{D} = \sum_{i=1}^{N} \tilde{D}_i) \to (Y, D)$ which is unramified over U so that for each $j = 1, \ldots, \ell$, one has*

- *either* $\mathrm{ord}_{\tilde{D}_j}(\mu^* D) \geq m$,
- *or the local monodromy group of $\mu^* \mathcal{L}$ around $\tilde{D}_j$ is trivial.* □

Let us now prove Theorem 6.1.

Proof of Theorem 6.1. By the proof of Theorem A, Y is a projective manifold. For the $(1, 1)$-form ω on U defined by

$$\omega = -i\mathrm{tr}(\theta_h^* \wedge \theta), \tag{6.2}$$

by Lemma 1.16, we know that ω is a pseudo-Kähler form whose holomorphic sectional curvature is bounded from above by $-\frac{1}{4^{\mathrm{rank}\, E-1}}$. Let ϖ be the positive closed $(1, 1)$-current on Y which is the trivial extension of ω. By Lemma 1.15, the class $\{\varpi\}$ is big and nef. Choose $\varepsilon > 0$ so that $\{\varpi\} - \varepsilon D$ is still big and

$$E_{nK}(\{\varpi\} - \varepsilon\{D\}) = E_{nK}(\{\varpi\}). \tag{6.3}$$

Pick $m \gg 0$ so that $m\varepsilon \geq 2^{2\mathrm{rank}\, E-1}\pi$. Let $\mathcal{L}$ be the local system relative to the tame harmonic bundle. By Lemma 6.2, we find a log morphism $\mu : (X, \tilde{D} = \sum_{i=1}^{N} \tilde{D}_i) \to (Y, D)$ from a smooth projective log pair $(X, \tilde{D})$ which is unramified over U satisfying the properties therein.

Set $D_X \subset \tilde{D}$ to be the sum of all $\tilde{D}_j$'s so that the local monodromy group of $\mu^* \mathcal{L}$ around $\tilde{D}_j$ is *not* trivial. Then by the dichotomy in Lemma 6.2, $\mu^* D - m D_X$ is an effective divisor, and the monodromy of $\mu^* \mathcal{L}$ around $\tilde{D}_i$ with $\tilde{D}_i \not\subset D_X$ is trivial. By Proposition 6.4 below, the pull-back harmonic bundle extends to a nilpotent harmonic bundle over $X - D_X$. Such a nilpotent harmonic bundle induces a pseudo-Kähler metric ω_2 on $X - D_X$. One has $\omega_2 = \mu^* \omega$ over $\tilde{U}$. ω_2 thus has non-positive holomorphic bisectional curvature and holomorphic sectional curvature bounded from above by $-\frac{1}{4^{\mathrm{rank}\, E-1}}$. Denote by ϖ_2 the closed positive current which is the trivial extension of ω_2.

Claim 6.3. $\mu^* \varpi = \varpi_2$.

Proof of Claim 6.3. By the very definition of trivial extension, we have

$$\mu^*\varpi \geq \mu^*\varpi - \varpi_2 \geq 0. \tag{6.4}$$

Pick any point $x \in \tilde{D}$, and choose admissible coordinates $(\Omega; x_1, \ldots, x_n)$ and $(\Omega_2; y_1, \ldots, y_n)$ around x and $y = \mu(x)$ with $\mu(\Omega_1) \subset \Omega_2$ so that $\Omega_1 \cap \tilde{D} = (x_1 \cdots x_{\ell_1} = 0)$ and $\Omega_2 \cap D = (y_1 \cdots y_{\ell_2} = 0)$. Since μ is a log morphism, one has $\mu^* y_i = g_i(x) \prod_{j=1}^{\ell_1} x_j^{a_{ij}}$ with $a_{ij} \in \mathbb{Z}_{\geq 0}$ and $g_i(x) \in \mathcal{O}(\Omega_1)$. By (1.3), the local potential ϕ of $\varpi = \mathrm{dd^c}\phi$ satisfies

$$\phi \gtrsim -\log\Big(\prod_{j=1}^{\ell_1}(-\log|y_j|^2)\Big).$$

Hence the local potential $\phi \circ \mu$ of $\mu^*\varpi = \mathrm{dd^c}\phi \circ \mu$ satisfies

$$\phi \circ \mu \gtrsim -\log\Big(\prod_{j=1}^{\ell_2}(-\log|x_j|^2)\Big).$$

Therefore, the Lelong numbers of $\mu^*\varpi$ are zero everywhere, and by (6.4), the same holds for the positive current $\mu^*\varpi - \varpi_2$. On the other hand, since $\omega_2 = \mu^*\omega$, $\mu^*\varpi - \varpi_2$ is thus supported on $\tilde{D}$. By the support theorem [Dem12b, (2.14) Corollary], $\mu^*\varpi - \varpi_2 = \sum_{i=1}^N \lambda_i[\tilde{D}_i]$ with $\lambda_i \geq 0$. Hence $\mu^*\varpi - \varpi_2 = 0$. □

Note that

$$\begin{aligned}\mu^*(\{\varpi\} - \varepsilon\{D\}) &= \{\varpi_2\} - \varepsilon\{\mu^*D - mD_X\} - m\varepsilon\{D_X\}\\ &= \{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\} - \varepsilon\{\mu^*D - mD_X\} - (m\varepsilon - 2^{2\operatorname{rank} E-1}\pi)\{D_X\}.\end{aligned}$$

Recall that $\mu^*D - mD_X \geq 0$, and $m\varepsilon \geq 2^{2\operatorname{rank} E-1}\pi$. Hence

$$\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\} = \mu^*(\{\varpi\} - \varepsilon\{D\}) + \{D'\}$$

where D' is some effective $\mathbb{R}$-divisor supported in $\tilde{D}$. Therefore, $\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\}$ is big with its non-Kähler locus

$$E_{nK}(\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\}) \subset E_{nK}(\mu^*(\{\varpi\} - \varepsilon\{D\})) \cup \tilde{D}.$$

Applying Lemma 6.5 below to $\{\varpi\} - \varepsilon\{D\}$, we obtain

$$E_{nK}(\mu^*(\{\varpi\} - \varepsilon\{D\})) \subset \mu^{-1}(E_{nK}(\{\varpi\} - \varepsilon\{D\})) \cup \tilde{D}.$$

By (6.3), one has

$$E_{nK}(\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\}) \subset \mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}. \tag{6.5}$$

Recall that the holomorphic sectional curvature of ω_2 is bounded from above by $-\frac{1}{4^{\operatorname{rank} E-1}}$. By Theorems 3.1.(ii) and 4.1.(ii), we conclude that X is both Picard hyperbolic and algebraically hyperbolic modulo $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$. Theorems 6.1.(ii) and 6.1.(iii) follows.

Let $\tilde{Z} \subset X$ be any irreducible closed subvariety which is not contained in $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$. Let $g : Z \to \tilde{Z}$ be a desingularization so that $D_Z := g^{-1}(D_X)$ is a simple normal crossing divisor. Applying Theorem 1.11 we can pick a Kähler current

$$S \in \{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\}$$

so that S is smooth outside $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$ by (6.5). Since $g(\tilde{Z})$ is not contained in $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$, the pull-back g^*S exists and is a closed positive $(1,1)$ current in $g^*\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi g^*\{D_X\}$. Hence $g^*\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi g^*\{D_X\}$ is pseudo effective.

Write $Z^\circ := Z - D_Z$. We claim that $\omega_3 := g^*\omega_2$ is strictly positive at one point of Z°, hence is a pseudo Kähler form on Z°. Or else,

$$\int_{\tilde{Z}} \{\varpi_2\}^{\dim \tilde{Z}} = \int_{Z^\circ} (g^*\omega_2)^{\dim Z} = 0,$$

which implies that $\tilde{Z} \in E_{nK}(\{\varpi_2\})$ by Theorem 1.12. Since

$$E_{nK}(\{\varpi_2\}) \subset E_{nK}(\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi\{D_X\}) \subset \mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D},$$

this contradicts with the assumption that $\tilde{Z} \subset X$ is not contained in $\mu^{-1}(E_{nK}(\{\varpi\})) \cup \tilde{D}$. Therefore, ω_3 is a pseudo Kähler form.

By the curvature decreasing property of submanifolds, we conclude that the holomorphic bisectional curvature of ω_3 is non-positive, and the holomorphic sectional curvature of ω_3 is bounded from above by $-\frac{1}{4^{\operatorname{rank} E-1}}$. Let ϖ_3 be the closed positive $(1,1)$ current on Z which is the trivial extension of ω_3. One can employ a similar method as for Claim 6.3 to show that

$$g^*\{\varpi_2\} = \{\varpi_3\}.$$

Recall that $g^*\{\varpi_2\} - 2^{2\operatorname{rank} E-1}\pi g^*\{D_X\}$ is pseudo effective. Since $g^*D_X \geq D_Z$, $\{\varpi_3\} - 2^{2\operatorname{rank} E-1}\pi\{D_Z\}$ is thus also pseudo effective. By Lemma 1.16, one has

$$\frac{1}{2^{2\operatorname{rank} E-1}\pi}\{\varpi_3\} \leq K_Z + D_Z.$$

Hence K_Z is big. Theorem 6.1.(i) follows.

Lastly, by (6.2), ω is strictly positive at any point where θ is injective. (6.1) then follows from (3.1). □

We state and prove the following crucial extension result for nilpotent tame harmonic bundles across the boundary components around which the local monodromies of the corresponding local system are trivial. Its proof was communicated to us by C. Simpson, and it uses the deep theorem by Mochizuki on the correspondence between tame pure imaginary harmonic bundles and semisimple local systems over quasi-projective manifolds.

Proposition 6.4. *Let Y be a projective manifold and let $D = \sum_{i=1}^m D_i$ be a simple normal crossing divisor on Y. Let (E, θ, h) be a nilpotent harmonic bundle on $U := Y - D$, whose corresponding complex local system is denoted by $\mathcal{L}$. Assume that for $i = 1, \ldots, r$ the local monodromy of $\mathcal{L}$ around the component D_i is trivial. Then (E, θ, h) extends to a nilpotent harmonic bundle on $U' := Y - \sum_{i=r+1}^m D_i$.*

Proof. Since θ is assumed to be nilpotent, the eigenvalue of the residue $\operatorname{Res}(\theta)$ at each component D_i is thus zero. Hence (E, θ, h) is a *tame pure imaginary harmonic bundle* in the sense of [Moc07, Definition 22.3]. By [Moc07, Proposition 22.15], $\mathcal{L}$ is semisimple. Hence it is a direct sum $\mathcal{L} := \oplus_\alpha \mathcal{L}_\alpha \otimes \mathbb{C}^{m_\alpha}$, where $\mathcal{L}_\alpha$ is a simple local system and $m_\alpha > 0$. Since the local monodromy of $\mathcal{L}$ around the component D_i is trivial for $i = 1, \ldots, r$, so is $\mathcal{L}_\alpha$ for each α. Hence $\mathcal{L}_\alpha$ extends to a local system $\mathcal{L}'_\alpha$ on U'. Since the map between fundamental groups $\pi_1(U) \to \pi_1(U')$ is surjective, $\mathcal{L}'_\alpha$ is thus also simple.

By [Moc07, Theorem 25.21], there is a tame pure imaginary harmonic bundle $(E_\alpha, \theta_\alpha, h_\alpha)$ on U whose corresponding local system is $\mathcal{L}_\alpha$. Moreover, by the uniqueness property of the correspondence between semisimple local systems and tame pure imaginary harmonic bundles proved in [Moc07, Theorem 25.28], one has

$$(E, \theta, h) = \oplus_\alpha \big(\oplus_{m_\alpha} (E_\alpha, \theta_\alpha), h_\alpha \otimes g_\alpha \big),$$

where g_α denotes a hermitian metric of $\mathbb{C}^{m_\alpha}$. Since $t^{\operatorname{rank} E} = \det(t-\theta) = \prod_\alpha \det(t-\theta_\alpha)^{m_\alpha}$, one has $\det(t - \theta_\alpha) = t^{\operatorname{rank} E_\alpha}$. Hence each $(E_\alpha, \theta_\alpha, h_\alpha)$ is a nilpotent harmonic bundle.

Again by [Moc07, Theorem 25.21], there is a tame pure imaginary harmonic bundle on $(E'_\alpha, \theta'_\alpha, h'_\alpha)$ on U' whose corresponding local system is $\mathcal{L}'_\alpha$. The restriction $(E'_\alpha, \theta'_\alpha, h'_\alpha)|_U$ is thus a tame pure imaginary harmonic bundle with the corresponding local system $\mathcal{L}'_\alpha|_U = \mathcal{L}_\alpha$. By the uniqueness result in [Moc07, Theorem 25.28], $(E'_\alpha, \theta'_\alpha)|_U = (E_\alpha, \theta_\alpha)$ and $c_\alpha \cdot h'_\alpha|_U = h_\alpha$ for some constant $c_\alpha > 0$. Since characteristic polynomial $\det(t -$

$\theta'_\alpha)|_U = \det(t-\theta_\alpha) = t^{\operatorname{rank} E_\alpha}$, by continuity $\det(t-\theta'_\alpha) = t^{\operatorname{rank} E_\alpha}$ on U'. Hence $(E'_\alpha, \theta'_\alpha, h'_\alpha)$ is also nilpotent. Therefore, the nilpotent harmonic bundle

$$(E', \theta', h') = \oplus_\alpha \big(\oplus_{m_\alpha} (E'_\alpha, \theta'_\alpha), c_\alpha \cdot h'_\alpha \otimes g_\alpha \big)$$

defined on U' extends (E, θ, h). The proposition is proved. □

The following result allows us to control non-Kähler locus on ramified covers.

Lemma 6.5. *Let $\mu : (X, \tilde{D}) \to (Y, D)$ be a log morphism between compact Kähler log pairs, which is unramified over $X - \tilde{D}$. Let α be a big class on Y. Then*

$$E_{nK}(\mu^*\alpha) \subset \mu^{-1}(E_{nK}(\alpha)) \cup \tilde{D} \tag{6.6}$$

Proof. By Theorem 1.11, one can take a Kähler current $T \in \alpha$ with analytic singularities which is smooth outside $E_{nK}(\alpha)$. Choose a Kähler form ω on Y so that $T \geq \omega$. Then $\{\mu^*\omega\}$ is a big and nef class, and by Theorem 1.12, one has

$$E_{nK}(\{\mu^*\omega\}) \subset \tilde{D}.$$

Applying Theorem 1.11 again, there is a global quasi-psh function φ on X with analytic singularities which is smooth outside $\tilde{D}$ so that $\mu^*\omega + dd^c\varphi$ is a Kähler current on X. It follows from $T \geq \omega$ that $\mu^*T + dd^c\varphi \geq \mu^*\omega + dd^c\varphi$. Hence $\mu^*T + dd^c\varphi$ is a Kähler current with analytic singularities, which is smooth outside $\mu^{-1}(E_{nK}(\alpha)) \cup \tilde{D}$. Since $\mu^*T + dd^c\varphi \in \mu^*\alpha$, by the very definition of non-Kähler locus Definition 1.10, one has

$$E_{nK}(\mu^*\alpha) \subset \mu^{-1}(E_{nK}(\alpha)) \cup \tilde{D}.$$

The lemma is proved. □

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